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II. RELATING TO THE DEFINITION OF A FUNCTION  $E$ .

By OSCAR SCHMIEDEL, Bellevue College, Bellevue, Nebraska.

A function  $E$  is defined by the following three statements:

$$1. E_{[r]}^{(l, n, x)} = 1 + \frac{x}{l} + \frac{x^2}{l(l+n)} + \cdots + \frac{x^r}{l(l+n)\cdots(l+r-1)n}.$$

$$2. E_{(l, n, x)}^{[r]} = -\frac{l-n}{x} - \frac{(l-n)(l-2n)}{x^2} - \cdots - \frac{(l-n)(l-2n)\cdots(l-r-1)n}{x^{r-1}}.$$

$$3. E^{(x^n)} = 1 + \frac{nx^n}{1+n} + \frac{(nx^n)^2}{1+n} + \frac{(nx^n)^3}{(1+n)(1+2n)} + \cdots$$

The second statement is a consequence from the first when  $r$  is negative; the third a special definition for the purpose of simplification in the present remarks. The index  $r$  in the first and second may be omitted when for particular values of  $l$  and  $n$  the series either terminates or becomes infinite; the index  $l$  may also be omitted when  $l = 1$ ; thus,  $E^{(1, x)} = e^x$ .

It will now appear by reference to the September, 1917, number of the MONTHLY, pp. 342-343, that<sup>1</sup>  $\int \sin \theta^2 d\theta$  and  $\int \cos \theta^2 d\theta$  may be written in the following concise forms:

$$\begin{aligned} \int \sin \theta^2 d\theta &= -\theta \sum_{i=0}^{\infty} \prod_{s=1}^{s=i} \frac{2\theta^2}{1+2s} \cdot \sin \left( t \frac{\pi}{2} - \theta^2 \right) \\ &= \theta \sin \theta^2 \left( 1 - \frac{(2\theta^2)^2}{3 \cdot 5} + \frac{(2\theta^2)^4}{3 \cdot 5 \cdot 7 \cdot 9} - + \cdots \right) \\ &\quad - \theta \cos \theta^2 \left( \frac{2\theta^2}{3} - \frac{(2\theta^2)^3}{3 \cdot 5 \cdot 7} + \frac{(2\theta^2)^5}{3 \cdot 5 \cdots 11} - + \cdots \right) \\ &= \frac{1}{4} i \theta^{-1} \sin \theta^2 (E^{(-i\theta^2)} - E^{(i\theta^2)}) + \frac{1}{4} \theta^{-1} \cos \theta^2 (E^{(-i\theta^2)} + E^{(i\theta^2)}) \\ &\quad - \frac{1}{2} \theta^{-1} \cos \theta^2 \\ &= \frac{1}{4} \theta^{-1} (\cos \theta^2 + i \sin \theta^2) E^{(-i\theta^2)} + \frac{1}{4} \theta^{-1} (\cos \theta^2 - i \sin \theta^2) E^{(i\theta^2)} \\ &\quad - \frac{1}{2} \theta^{-1} \cos \theta^2 \\ &= \frac{1}{4} \theta^{-1} (e^{i\theta^2} E^{(-i\theta^2)} + e^{-i\theta^2} E^{(i\theta^2)}) - \frac{1}{2} \theta^{-1} \cos \theta^2; \\ \int \cos \theta^2 d\theta &= \frac{1}{4} i \theta^{-1} (e^{i\theta^2} E^{(-i\theta^2)} - e^{-i\theta^2} E^{(i\theta^2)}) + \frac{1}{2} \theta^{-1} \sin \theta^2. \end{aligned}$$

If in a similar manner the integral of  $e^{\theta^2} d\theta$  is sought by means of the reduction

<sup>1</sup> We write  $\sin \theta^2$  for  $\sin (\theta^2)$ , etc.

formula

$$\int \theta^m e^{\theta^n} d\theta = \frac{1}{n} \theta^{m-n+1} e^{\theta^n} - \frac{m-n+1}{n} \int \theta^{m-n} e^{\theta^n} d\theta,$$

the following two forms are obtained:

$$\begin{aligned} \int \theta^m e^{\theta^n} d\theta &= \frac{1}{n} \theta^{m-n+1} e^{\theta^n} \sum_{t=0}^{t=r} \prod_{s=1}^{s=t} \frac{sn-m-1}{n\theta^n} + \prod_{s=1}^{s=r+1} \frac{sn-m-1}{n} \cdot \int \theta^{m-(r+1)n} e^{\theta^n} d\theta, \\ \int \theta^m e^{\theta^n} d\theta &= -\frac{1}{n} \theta^{m-n+1} e^{\theta^n} \sum_{t=1-r}^{t=-1} \prod_{s=t+1}^{s=0} \frac{n\theta^n}{sn-m-1} \\ &\quad + \prod_{s=2-r}^{s=0} \frac{n}{sn-m-1} \cdot \int \theta^{m+(r-1)n} e^{\theta^n} d\theta, \end{aligned}$$

where  $r$  is a positive integer.

The second of these gives, for  $m = 0$ ,  $n = 2$ , the desired development as a convergent series, which by 3. takes the form

$$\begin{aligned} \int e^{\theta^2} d\theta &= -\frac{1}{2}\theta^{-1}e^{\theta^2} \left( -\frac{2\theta^2}{1} + \frac{(2\theta^2)^2}{3} - \frac{(2\theta^2)^3}{3 \cdot 5} + \dots \right) \\ &= -\frac{1}{2}\theta^{-1}e^{\theta^2} E(-\theta^2) + \frac{1}{2}\theta^{-1}e^{\theta^2}. \end{aligned}$$

Comparison of these three results shows the following relation to hold:

$$\int \cos \theta^2 d\theta + i \int \sin \theta^2 d\theta = \int e^{i\theta^2} d\theta,$$

as was to be expected.

When  $m+1$  is a multiple of  $n$ , the development of  $\int \theta^m e^{\theta^n} d\theta$  is a finite series. Thus, let  $m = 7, 0$ ;  $n = 2, \frac{1}{4}$ , respectively, and  $r = 3$ , and the first formula gives:

$$\begin{aligned} \int \theta^7 e^{\theta^2} d\theta &= \frac{1}{2}\theta^6 e^{\theta^2} \sum_{t=0}^{t=3} \prod_{s=1}^{s=t} (s-4)\theta^{-2} \\ &= -3e^{\theta^2} \left( 1 - \frac{\theta^2}{1!} + \frac{\theta^4}{2!} - \frac{\theta^6}{3!} \right) \\ &= \frac{1}{8}\theta^8 e^{\theta^2} E_{(1/5, -1/5\theta^2)} \end{aligned} \quad [\text{by 2}];$$

$$\begin{aligned} \int e^{\theta^{1/4}} d\theta &= 4\theta^{3/4} e^{\theta^{1/4}} \sum_{t=0}^{t=3} \prod_{s=1}^{s=t} (s-4)\theta^{-1/4} \\ &= -4! e^{\theta^{1/4}} \left( 1 - \frac{\theta^{1/4}}{1!} + \frac{\theta^{2/4}}{2!} - \frac{\theta^{3/4}}{3!} \right) \\ &= \theta e^{\theta^{1/4}} E_{(1/5, -1/5\theta^{1/4})} \end{aligned} \quad [\text{by 2}].$$

It is interesting to observe that these values of  $m$  and  $n$  substituted in the second of the formulae above will give, after a little transformation, the same finite

forms; thus, in the latter case:

$$\int e^{\theta^{1/4}} d\theta = -4! e^{\theta^{1/4}} \left( 1 - \frac{\theta^{1/4}}{1} + \frac{\theta^{2/4}}{2!} - \frac{\theta^{3/4}}{3!} \right) + 4!,$$

differing from the previous result only by a constant term.

This result is general, and gives, when the exponent of the argument is a unit fraction:

$$\int e^{\theta^{1/n}} d\theta = (-1)^{n-1} n! e^{\theta^{1/n}} \sum_{t=0}^{t=n-1} (-1)^t \frac{\theta^{t/n}}{t!},$$

or, the same expression plus the term  $(-1)^n n!$ .

### III. RELATING TO MAGIC SQUARES FOR THE NEW YEAR, 1918.

By S. A. COREY, Albia, Iowa.

The two following  $4 \times 4$  magic squares are taken indiscriminately one from each of two large families of  $4 \times 4$  magic squares with sum 1918 which may be formed in like manner from the two sets of 15 square numbers here employed. It is improbable that other families of  $4 \times 4$  magic squares with the sum 1918 can be found with similar characteristics.

464	383	513	558
591	480	482	365
414	429	527	548
449	626	396	447

(A)

461	409	516	532
565	483	508	362
440	426	501	551
452	600	393	473

(B)

Some of the peculiarities of the *families* of magic squares from which (A) and (B) are taken are as follows:

Each number employed may be decomposed into the sum of the squares of *four* separate numbers taken from the series of natural numbers, 1 to 18 inclusive (18 being the last two figures of 1918).

In forming such sums *all* the square numbers used are used just *four* times with *one* exception in each case as follows:

In (A) 10 is used *eight* times, and

In (B) 6 is used *eight* times.

In forming the numbers employed in (A) all the natural numbers, 1 to 18 inclusive, are used, except 1, 11 and 13, the sum of which is 25.

In forming the numbers employed in (B) all the natural numbers, 1 to 18 inclusive, are used, except 5, 9 and 11, the sum of which is likewise 25. To illustrate:

$$2^2 + 6^2 + 10^2 + 18^2 = 464, 3^2 + 7^2 + 10^2 + 15^2 = 383, 4^2 + 8^2 + 12^2 + 17^2 = 513, 5^2 + 9^2 + 14^2 + 16^2 = 558 [464, 383, 513 \text{ and } 558 \text{ being the numbers in}$$